PMM U.S.S.R.,Vol.49,No.3,pp.273-281,1985
Printed in Great Britain

0021-8928/85 $10.00+0.00
Pergamon Journals Ltd.

THE STABILITY OF MULTIDIMENSIONAL HAMILTONIAN SYSTEMS*

V.N. TKHAI

The Lyapunov stability of the zero solution of autonomous periodic Hamiltonian systems is investigated. It is assumed that the linearized system is stable, its matrix can be reduced to diagonal form, and all characteristic indices (or roots of characteristic equations) are purely imaginary. The problem is completely solved for periodic systems with one degree of freedom, and for autonomous systems with two degrees of freedom /1-7/. The conditions of stability for systems with an arbitrary number of degrees of freedom are obtained below. The results are used to solve the problem of the Lyapunov stability of triangular libration points of the restricted three-body problem (plane elliptic and three-dimensional circular).

1. Statement of the problem. Consider the problem of the Layapunov stability of the zeroth solution of the canonical system with an analytic Hamiltonian function $H(\mathbf{p}, \mathbf{q}, t)$, 2π periodic in time t (or not explicitly dependent on time)

$$\frac{d\mathbf{p}}{dt} = -\frac{\partial H}{\partial \mathbf{q}}, \quad \frac{d\mathbf{q}}{dt} = \frac{\partial H}{\partial \mathbf{p}}, \quad H(0, 0, t) \equiv 0 \tag{1.1}$$

$$\mathbf{p} = (p_1, \ldots, p_n), \ \mathbf{q} = (q_1, \ldots, q_n)$$

We will assume that the linearized system is stable, all its characteristic indices (or roots of the characteristic equation) are purely imaginary and different, and that the system has no third- and fourth-order resonances. We shall investigage the autonomous problem, when the Hamiltonian H is not a function of fixed sign (when it is, the stability problem is solved by the Lagrange-Dirichlet theorem).

With these assumptions the Hamiltonian of the system reduces to the form /8,9/

$$H = \sum_{i=1}^{n} \lambda_i r_i + \sum_{i,j=1}^{n} c_{ij} r_i r_j + H^{(1)}(\mathbf{p}, \mathbf{q}, t) \tag{1.2}$$

$$(2r_i = p_i^2 + q_i^2; \ \lambda_i, \ c_{ij} = \text{const}; \ \lambda_i \neq 0)$$

where the function $H^{(1)}(\mathbf{p}, \mathbf{q}, t)$ contains terms $\mathbf{p}, \mathbf{q}$ of order not lower than the fifth. The coefficients c_{ij} in (1.2) in terms of coefficients of second- and fourth-order forms in the expansion of the initial Hamiltonian in series in $\mathbf{p}, \mathbf{q}$ were calculated in /9/.

As in /1-4/, problem (1.1), (1.2) has, for $n = 1$, a periodic system, and for $n = 2$ in the autonomous case, has a positive solution in the sense of stability, if $c_{11} \neq 0$ and $c_{11}\lambda_2^2 - 2c_{12}\lambda_1\lambda_2 + c_{22}\lambda_1^2 \neq 0$, respectively. The problem of Lyapunov stability for $n \geqslant 2$ in the periodic case and for $n \geqslant 3$ in the autonomous case has so far not been solved.

2. Instability. Hamiltonian systems are special cases of systems with invariant measure. A simple statement can be shown to hold for these systems, which in spite of its simplicity, sheds light on Lyapunov instability in the neighbourhood of the singular point. Since we are seeking in the first instance a solution of problem (1.1), (1.2), we shall limit the analysis to a 2π periodic or autonomous system specified by the differential equations

$$d\mathbf{x}/dt = \mathbf{X}(\mathbf{x}, t), \quad \mathbf{X}(0, t) \equiv 0, \quad \operatorname{div} \mathbf{X} = 0 \tag{2.1}$$

$$\mathbf{x} = (x_1, \ldots, x_L), \ \mathbf{X} = (X_1, \ldots, X_L)$$

It is assumed that the right sides of (2.1) ensure the existence, uniqueness, and continuous dependence of solutions on the initial conditions in some neighbourhood of zero.

We shall say that the property S is satisfied by system (2.1), if $\varepsilon > 0$ can be found that, no matter how small $\delta > 0$, a point will be found in the δ-neighbourhood such that a trajectory beginning at that point leaves the neighbourhood of ε in a finite time for increasing as well as decreasing time t.

*Prikl. Matem. Mekhan. 49,3,355-365,1985

Lemma. The property of Lyapunov instability of the zero solution of system (2.1) for $0 \leqslant t < +\infty$ or $-\infty < t \leqslant 0$ is equivalent to the property S.

Proof. Suppose that the zero solution of the periodic system (2.1) is Lyapunov unstable when $0 \leqslant t < +\infty$. Then, in an arbitrary small δ-neighbourhood of zero there is at least one point which, moving according to (2.1), leaves that ε-neighbourhood in finite time. By virtue of the continuous dependence on the initial condition a certain set G of points of non-zero measure belonging to the δ-neighbourhood will also leave the ε-neighbourhood after a finite time.

As we are now interested in the previous history of the set G we shall consider Eq.(2.1) when the time t is reduced.

When t is changed by 2π, (2.1) defines a one-to-one continuous mapping T of the "phase" space $(\mathbf{X})$ into itself. That mapping maintains the phase volume. Let $T^k G$ $(k = 0, \pm 1, \pm 2, \ldots)$ be the image of the set G after n-tuple mapping T. By virtue of the instability the set $T^{k_*} G$ for some k_* lies outside the ε-neighbourhood. If when t is reduced all trajectories beginning in G, belong to the ε-neighbourhood, then $T^{k_* + k}G$ $(k = -k_*, -2k_*, -3k_*, \ldots)$ also belongs to that neighbourhood. But this is impossible by virtue of the Poincaré theorem on return /10/, since the measure of G is non-zero, while the measure of the ε-neighbourhood is finite. Moreover, almost all points of G, in the meaning of measure, must leave the ε-neighbourhood.

Thus the set G contains at least one point such that the trajectory beginning at that point will leave the ε-neighbourhood in a finite time for increasing as well as decreasing t. Obviously, such a point also exists for an instability in $-\infty < t \leqslant 0$, and also for an unstable autonomous system.

3. Basic idea of the proof of the theorem on the stability of system (1.1), (1.2).

To use indirect proof we introduce an integral relation which will be fundamental in obtaining the respective contradiction. In the case of a periodic system a contradiction is obtained, when the quadratic form

$$H_4 = \sum_{i,j=1}^{n} c_{ij} r_i r_j \tag{3.1}$$

is of fixed sign in the positive cone $r_i \geqslant 0$ $(i = 1, \ldots, n)$.

Let us consider an unstable periodic system with the Hamiltonian function H_4 of fixed sign. We define the γ-neighbourhood of zero, by the inequality $|H_4| \leqslant \gamma^4$.

Due to the equation of motion (1.1), (1.2) the derivative $H_4^{\cdot}$ of the function H_4 divides the phase space $(\mathbf{p}, \mathbf{q})$ for every t into three regions: $H_4^{\cdot} < 0, H_4^{\cdot} = 0, H_4^{\cdot} > 0$. By virtue of maintaining the phase volume, and the assumption on instability, each of these regions is non-empty. For all points of the surface $|H_4| = \gamma^4$ of region $H_4 H_4^{\cdot} < 0$, the integral curves enter the inside of that surface as t increases.

We denote by $S_1(\gamma)$ the curve defined on the surface $|H_4| = \gamma^4$ as follows. The inequality $H_4 H_4^{\cdot} \leqslant 0$ holds on $S_1(\gamma)$, and $H_4^{\cdot} = 0$ is only at one of the ends of the curve $S_1(\gamma)$ at the point $C(\gamma)$. All integral curves that pass through $S_1(\gamma)^1$ * (*Except the curve passing through $C(\gamma)$) and enter the region $|H_4| < \gamma^4$, leave it after a finite time. The integral curve $S(\gamma)$ passes through the other end of curve $S_1(\gamma)$, the point $A(\gamma)$.

The integral curve $S(\gamma)$ intersects the surface $|H_4| = \gamma^4$ at the points $A(\gamma)$ and $B(\gamma)$, and the condition $\tau_1 > \tau/2$ is satisfied on it. Here τ is the time of motion from point $A(\gamma)$ to point $B(\gamma)$ and $S(\gamma)$, and τ_1 is the time of motion in the region $|H_4| \leqslant \gamma^4/4$.

The curve $S(\gamma)$ always exists for any $\gamma (0 < \gamma \leqslant \varepsilon)$. Indeed, the assumption of instablity implies that property S is satisfied. Hence the integral curve which enters region $|H_4| < \gamma^4$, passes through the δ-neighbourhood of zero, and leaves the region $|H_4| < \gamma^4$ after a finite time, does necessarily exist. The number δ may be arbitrarily small, and besides, the derivative of the function $r = r_1 + \ldots + r_n$ by virtue of (1.1), (1.2) is of order r^{3}. Hence the smaller δ the larger τ_1 will be for a given γ.

The curve $S_1(\gamma)$ may also be constructed for some $\gamma (0 < \gamma \leqslant \varepsilon)$ in accordance with the property S.

Let us denote by $Q(\varepsilon)$ the connected set of points on the surface $|H_4| = \varepsilon^4$ on which, when $t = 0$, the trajectories, which enter and leave the region $|H_4| < \varepsilon^4$ after a finite time, begin. Now suppose that it proved impossible to construct $S_1(\gamma)$ when $\gamma = \varepsilon$. Then each set $Q(\varepsilon)$ containing points $A_i(\varepsilon)$ has no boundary points where $H_4^{\cdot} = 0$. Let us consider $Q_1(\varepsilon)$, the arbitrary closed connected set of point belonging to $Q(\varepsilon)$. The trajectories beginning on $Q_1(\varepsilon)$ provide for some γ on the surface $|H_4| = \gamma^4$ a connected closed set $Q_1(\gamma)$ that contains the points $H_4^{\cdot} = 0$. The construction of the curve $S_1(\gamma) \subset Q_1(\gamma)$ will thus be completed, if $Q_1(\varepsilon)$ is selected so that $Q_1(\gamma)$ contains the point $A(\gamma)$. This is always possible by virtue of property S.

The trajectories that pass through $S_1(\gamma)$, with subsequent intersection of the surface

$|H_4| = \gamma^4$, yield the curve $S_2(\gamma)$ whose ends are at the points $B(\gamma)$ and $C(\gamma)$. Curves $S(\gamma)$, $S_1(\gamma)$ and $S_2(\gamma)$ belong to a two-dimensional surface in extended phase space, and that surface is produced by the integral lines of the 1-form $\mathbf{p}\, d\mathbf{q} - Hdt$. Hence with the appropriate choice of integration direction from the point $A(\gamma)$ to the point $B(\gamma)$, we have

$$\int_{S(\gamma)} \mathbf{p}\, d\mathbf{q} - H\, dt = \int_{S_1(\gamma)+S_2(\gamma)} \mathbf{p}\, d\mathbf{q} - H\, dt \tag{3.2}$$

Changing the scale by replacing the variables p, q for $p\gamma, q\gamma$, we shall subsequently consider the Hamiltonian, which depends on the small parameter γ

$$H = \sum_{i=1}^{n} \lambda_i r_i + \gamma^2 H_4 + \gamma^3 H^{(1)}(\mathbf{p}, \mathbf{q}, \gamma, t) \tag{3.3}$$

For a system with the Hamiltonaian (3.3) curves $S(\gamma)$, $S_1(\gamma)$ and $S_2(\gamma)$ are defined on the surface $|H_4| = 1$. Omitting the index γ in the notation of these curves, we shall use instead of relation (3.2) the equivalent relation

$$\int_{S} \mathbf{p}\, d\mathbf{q} - \mathbf{q}\, d\,\mathbf{p} - 2H\, dt = \int_{S_1+S_2} \mathbf{p}\, d\mathbf{q} - \mathbf{q}\, d\mathbf{p} - 2H\, dt \tag{3.4}$$

4. **Stability.** Let the curve S_ω be defined on the surface $|H_4| = \omega^4$ (ω — const, $\omega \gg 1$), when $t = 0$, such that the integral curves beginning on S_ω yield on the surface $|H_4| = 1$ the curve S_1. We shall consider the two-dimensional manifold of initial conditions σ_0 that contain the curve S_ω, when $t = 0$. The trajectories that begin on σ_0 determine for every t the two-dimensional manifold σ_t and in a finite time interval belong to the three-dimensional integral manifold Σ in the extended phase space. For each value of the parameter t we have /11/

$$\sum_{i=1}^{n} (p_i\, dq_i - q_i\, dp_i) = 2\eta\, d\xi$$

where the variables ξ, η are mutually independent on Σ, if they are independent on σ_0*. (*In thise case the problem of reduction consists of determining the integrating multiplier for the equation $M(x, y)\, dx + N(x, y)\, dy = 0$ which is specified in a restricted closed region of the plane). In the latter case we obtain on Σ the equation

$$\sum_{i=1}^{n} (p_i\, dq_i - q_i\, dp_i) - 2H\, dt = 2(\eta\, d\xi - \Gamma\, dt) \tag{4.1}$$

where the function Γ depends on ξ, η, t and γ. By the same token a canonical system with one degree of freedom and the Hamiltonian Γ is constructed on Σ in the variables ξ, η.

We select σ_0 so as to have the independent variables R, φ defined on σ_0

$$\sum_{i=1}^{n} (p_i\, dq_i - q_i\, dp_i) = 2R\, d\varphi, \quad R^2 = |H_4|$$

Such a selection of σ_0 is always possible in an infinite number of ways.

On trajectories of the input system belonging to Σ we have

$$\frac{dR}{dt} = -\frac{\partial\Gamma}{\partial\varphi}, \quad \frac{d\varphi}{dt} = \frac{\partial\Gamma}{\partial R}$$

The first of these equations can also be obtained by differentiating R by virtue of the initial Eqs. (1.1) and (3.3). We have

$$\partial\Gamma / \partial\varphi = \gamma^3 R^{1/2} F(R, \varphi, \gamma, t)$$

where the function F does not exceed in absolute value some constant M which is independent of γ in the region $R \leqslant \omega^2$.

To determine $\partial\Gamma / \partial R$ we use the equation

$$\sum_{i=1}^{n} \left(p_i \frac{\partial H}{\partial p_i} + q_i \frac{\partial H}{\partial q_i}\right) - 2H = 2\left(R\frac{\partial\Gamma}{\partial R} - \Gamma\right)$$

obtained from (4.1) and valid along the trajectories of the system. We shall consider it for each (φ, t) as a differential equation in R and, taking into account the expression for $\partial\Gamma/\partial\varphi$, we obtain

$$\Gamma = a(\gamma, t)\, R \pm \gamma^2 R^2 + \gamma^3 R \Gamma_1(R, \varphi, \gamma, t)$$

where the function Γ_1 in region $R \leqslant \omega^2$ is limited in modulus by the finite number M_1, and the sign of $\gamma^2 R^2$ is the same as the sign of the quadratic form (3.1). It is obvious that it is always possible to assume $a(\gamma, t) \equiv 0$ since otherwise the canonical transformation $(R, \varphi) \to (R, \psi); \psi = \varphi - \int a(\gamma, t)\, dt$ reduces Γ to the form in which $a(\gamma, t) \equiv 0$. Hence along the trajectories

$$\partial\Gamma/\partial R = \pm 2\gamma^2 R + \gamma^3 \Phi(R, \varphi, \gamma, t), \quad |\Phi| < M_2$$

where the constants M_2, as well as M_1 are independent of γ.

It is now possible to evaluate the integrals in (3.4). Along the trajectory S we have

$$\Delta\varphi = \int_{t_1}^{t_1+\tau} d\varphi = \pm 2\gamma^2 \int_{t_1}^{t_1+\tau} R(t)\, dt + \gamma^3\Phi_1, \quad |\Phi_1| < M_2\tau$$

where $R(t)$ is the value of the variable R along the trajectory S, and t_1 and $t_1+\tau$ are values of t at points A and B, respectively. Hence

$$\int_{S_1+S_2} \sum_{i=1}^{n} (p_i\, dq_i - q_i\, dp_i) - 2H\, dt = 2\gamma^2 \Big[\pm \int_{t_1}^{t_1+\tau} (2R(t) - 1)\, dt\Big] + \gamma^3 V_1$$

$$\int_{S} \sum_{i=1}^{n} (p_i\, dq_i - q_i\, dp_i) - 2H\, dt = \pm 2\gamma^2 \int_{t_1}^{t_1+\tau} R^2(t)\, dt + \gamma^3 V_2, \quad |V_{1,2}| < M_3\tau$$

where the finite number M_3 is independent of γ, τ.

After these calculations, (3.4) can be rewritten in the form

$$\int_{t_1}^{t_1+\tau} [1 - R(t)]^2\, dt = \gamma V, \quad |V| < M_3\tau$$

Hence, in view of the choice of S we finally obtain

$$\frac{\tau}{8} < \int_{t_1}^{t_1+\tau} [1 - R(t)]^2\, dt < \gamma M_3 \tau$$

Since the constant M_3 is finite and independnet of γ, τ, the double inequality obtained cannot be satisfied for a fairly small γ.

The contradiction obtained proves that, when the quadratic form (3.1) is of fixed sign, the system cannot be unstable. Hence the following theorem holds.

Theorem 1. The periodic system (1.1), (1.2) is Lyapunov unstable, if the quadratic form (3.1) is of fixed sign in the positive cone $r_i \geqslant 0$ $(i = 1, \ldots, n)$.

Remarks. 1°. It follows from the proof that the condition of fixed sign of quadratic form (3.1) ensures the Lyapunov stability along the whole numerical axis $-\infty < t < +\infty$ (permanent stability /8/).

2°. From this there follows the conclusion of the Arnold-Moser theorem /1-4/ for systems with one degree of freedom $(n = 1)$.

5. The autonomous system. Obviously the quadratic form (3.1) of fixed sign also ensures the stability of an autonomous system. However, the result can be amplified in this case. Indeed, an autonomous system admits of the energy integral $H = h = \text{const}$ and the curves S, S_1 and S_2 can be selected on the integral surface $H = h$. If the curve S contains points from the δ-neighbourhood of zero, then for the respective integral surface we have $h \sim \delta$. The number δ can always be assumed to be smaller than γ^3 and we can construct on the integral surface a periodic system with $n-1$ degrees of freedom. All estimates in Sect.4 were made with an accuracy to terms in γ^3. Hence taking into account the relation $H = h$ $(|h| \leqslant \gamma^3)$, we find that the condition of stability is the fixed-sign form of the quadratic form (3.1) on the linear manifold.

$$\sum_{i=1}^{n} \lambda_i r_i = 0 \tag{5.1}$$

The proof may be obtained without constructing the respective periodic system with $n-1$ degrees of freedom while remaining within the limits of the reasoning of Sects. 3 and 4.

Indeed, with the condition that the quadratic form (3.1) be of fixed sign on the linear manifold (5.1), the curve S_1 and the manifold σ_0 can be selected in the region $|h| \leqslant \gamma^3$. Then the following estimate

$$\Big|\sum_{i=1}^{n} \lambda_i r_i\Big| < \gamma^2 \Big|\sum_{i,j=1}^{n} c_{ij} r_i r_j\Big| + \beta\gamma^3 \tag{5.2}$$

holds on Σ, where the finite number β independent of γ and h.

The formula

$$\Lambda=\sum_{i,j=1}^{n} c_{ij}r_ir_j \pm \varkappa\left(\sum_{i=1}^{n}\lambda_i r_i\right)^2$$

(the sign infront of $\varkappa$ is the same as that of the quadratic form (3.1) under condition (5.1) and when $r\neq 0$; $\varkappa$ is a properly selected to be constant) in the region $|h|\leqslant\gamma^3$ does not become zero when $r\neq 0$. Hence σ_0 can be selected so that the independent variables R, φ on Σ are defined as follows:

$$\sum_{i=1}^{n}(p_i dq_i - q_i dp_i) = 2R d\varphi, \quad R^2 = |\Lambda|$$

The Hamiltonian function can now be written in the form

$$H=\sum_{i=1}^{n}\lambda_i r_i \pm \gamma^2 R^2 \mp \varkappa\gamma^2\left(\sum_{i=1}^{n}\lambda_i r_i\right)^2+\gamma^3 H^{(1)}(\mathbf{p},\mathbf{q},\gamma,t)$$

Now taking the estimate (5.2) into account, the function Γ is determined in region $|h|\leqslant\gamma^3$ on the manifold Σ as in Sect. 4. Further considerations, that are also independent of the specific value of the constant energy, repeat the reasoning of Sect. 4, if the curves S_1 and S_2 were selected on the surface $R=1$ in the region $|h|\leqslant\gamma^3$.

Let us formulate the result obtained so far.

Theorem 2. The autonomous system (1.1), (1.2) is Lyapunov stable, if the system of equations

$$\sum_{i=1}^{n}\lambda_i r_i = 0, \quad \sum_{i,j=1}^{n} c_{ij}r_ir_j = 0, \quad r_i\geqslant 0 \quad (i=1,\ldots,n) \tag{5.3}$$

has no other solution than a trivial one.

Remarks. 1°. The application of Theorem 2 to investigate the stability of the steady motion of a mechanical system with cyclic (ignorable) coordinates ensures the absolute stability, since the proof mentioned above is independent of the specific value Δc_α ($\alpha=n+1,\ldots,N$) of the perturbation of cyclic coordinates; one has only to bear in mind that $|\Delta c_\alpha|\leqslant\gamma^3$.

2°. When $n=2$ the stability condition has the form

$$c_{11}\lambda_2^2 - 2c_{12}\lambda_1\lambda_2 + c_{22}\lambda_1^2 \neq 0$$

which is the same as the condition of the Arnold-Moser theorem /1-4/.

6. Some generalizations.

The method used to prove Theorems 1 and 2 enables us to extend these theorems to canonical systems with a non-analytic function H. The respective conditions for H are easily derived. On the other hand, the conditions of Lyapunov stability can also be obtained for an analytic function H, when the conditions of Theorems 1 and 2 are satisfied. For instance, when there are no resonances in the system up to the $2l_*$-th order, the Hamiltonian can be reduced to the form /8, 9/

$$H=\sum_{i=1}^{n}\lambda_i r_i + \sum_{l=2}^{l_*} H_{2l} + H^{(1)}(\mathbf{p},\mathbf{q},t)$$

$$H_{2l}=\sum_{l_1+\ldots+l_n=l} c_{l_1\ldots l_n} r_1^{l_1}\ldots r_n^{l_n}, \quad H^{(1)}=O\left(\left(\sum_{i=1}^{n} r_i\right)^{l_*+1/2}\right)$$

$$(2r_i=p_i^2+q_i^2;\ \lambda_i, c_{l_1\ldots l_n} - \text{const};\ \lambda_i\neq 0)$$

when the conditions of stability become:

for a periodic system we have the fixed-sign condition

$$H^{(0)}=\sum_{l=2}^{l_*}(l-1)H_{2l}$$

and for the autonomous we have the fixed-sign condition of $H^{(0)}$ on the linear manifold (5.1).

Theorems 1 and 2 are special cases of these statements when $l_*=2$.

A further generalization is connected with the investigation when the following resonances are present:

$$m_1\lambda_1+\ldots+m_n\lambda_n=m_*, \quad |m_1|+\ldots+|m_n|=m \tag{6.1}$$

where m_i, m_* are integers $m_*=0$ for the autonomous problem, and the number m is the order of the resonance /12/. Note that for resonances of order higher than the fourth, the Hamiltonian of the system reduces to the form (1.2) /9,12/ and the problem of stability is solved by Theorems 1 and 2. Hence only resonances of the second to fourth order require additional consideration.

For third-order resonances, the third-power resonance H_3^* in the normalized Hamiltonian is generally non-zero. However when in a series of numbers $m_1, \ldots, m_n$ there is no change of sign, the condition $H_3^* \not\equiv 0$ leads to Lyapunov instability /9,12/. When condition $H_3^* \equiv 0$ is satisfied, the problem of stability is solved, as above, by Theorems 1 and 2.

For a resonance of the second or fourth order the resonance form of the fourth power H_4^* in the normalized Hamiltonian contains besides terms of identical resonance (3.1) additional resonance terms. The problem of stability in these cases is solved by Theorems 1 and 2 independently of the specific form of these terms, if in their formulation we substitute H_4^* for the function H_4.

7. The fourth-order resonance. Let us obtain the sufficient conditions of stability for coefficients of the form H_4^* for the fourth-order resonances. In this case we have /9,12/

$$H_4^* = H_4(r_1, \ldots, r_n) + gz(r_1, \ldots, r_n)\cos\nu$$
$$z(r_1, \ldots, r_n) = r_1^{|m_1|/2} \ldots r_n^{|m_n|/2}, \quad \nu = m_1\varphi_1 + \ldots + m_n\varphi_n - m_*t \tag{7.1}$$

where g is a constant, $r_i, \varphi_i\ (i = 1, \ldots, n)$ are polar coordinates, and the function $H_4^*(r_1, \ldots, r_n)$ is determined by (3.1).

The Lyapunov instability conditions in this problem have the form /9,12/

$$|g|\, m_1^{m_1/2} \ldots m_n^{m_n/2} > |H_4(m_1, \ldots, m_n)|, \; m_i \geqslant 0$$
$$(i = 1, \ldots, n)$$

When this inequality is not satisfied the sufficient Lyapunov conditions can be derived using the generalization of Theorems 1 and 2. The derivation of such conditions generally requires the analysis of the fixed-sign property of the homogeneous fourth-order function H_4^* of the variables $p_i, q_i\ (i = 1, \ldots, n)$. However, in this problem the function H_4^* has a quite definite form, and depends on $n + 1$ variables $r_1, \ldots, r_n, \nu$. The sense of these variables implies that the problem consists of obtaining the conditions for the function H_4^* to be of fixed sign relative to the variables $r_1, \ldots, r_n$ in the positive cone.

It will be seen that the necessary condition for the function H_4^* to be of fixed sign is for the quadratic form H_4 to be of fixed sign. Let us assume that H_4 is of fixed sign. Then the following lemma holds.

Lemma. The function H_4^* is positive (negative) definite, if and only if the function

$$H_4^c = H_4 - |g|z \quad (H_4^0 = H_4 + |g|z)$$

Proof. From the inequalities valid in the positive cone when $r \neq 0$

$$H_4 > 0 \quad (H_4 < 0), \quad |g| \geqslant g\cos\nu, \quad z \geqslant 0$$

there follows the positive (negative) definiteness of the function H_4^* when the function H_4°, is positive (negative) definite. The converse statement follows from the fact that, when H_4^* is of fixed sign, the function H_4^c can only be of fixed sign since $H_4^* = H_4^\circ$ on the set $g\cos\nu = -|g|\,(g\cos\nu = |g|)$.

The lemma is quite useful in obtaining the sufficient conditions of stability of the coefficients $c_{ij}\ (i, j = 1, \ldots, n)$ and g of the form H_4^*. This is important for solving specific mechanical problems. Thus in the case of resonances $4\lambda_1 = m_*$ and $2(\lambda_1 \pm \lambda_2) = m_*$ the lemma immediately reduces the problem of the fixed-sign form of H_4^* to the respective problem of the quadratic form of the variables $r_1, \ldots, r_n$ in the positive cone.

Below, we shall investigate periodic systems with two degrees of freedom, and of autonomous systems with three degrees of freedom. For simplicity, we shall consider only the case when in a set of numbers $m_1, \ldots, m_n$ there is no change of sign. Moreover, we confine ourselves to obtaining solely the conditions of positive definiteness of the function H_4^*. Note that the case when m_i and m_j are of opposite sign $(m_im_j < 0)$ is investigated similarly.

A periodic system with two degrees of freedom. Fourth-order resonances

$$4\lambda_1 = m_*, \quad 2(\lambda_1 + \lambda_2) = m_*, \quad \lambda_1 + 3\lambda_2 = m_*$$

are possible in this system.

According to the lemma the problem of the positive definiteness of the function H_4^* for the first two resonances reduces to the positive definiteness of the quadratic form

$$H_4^c = b_{11}r_1^2 - 2b_{12}r_1r_2 + b_{22}r_2^2 \tag{7.2}$$

in the positive cone. For the resonance $4\lambda_1 = m_*$ we have

$$b_{11} = c_{11} - |g|, \quad b_{12} = c_{12}, \quad b_{22} = c_{22}$$

and for resonance $2(\lambda_1 + \lambda_2) = m_*$

$$b_{11} = c_{11}, \quad b_{12} = c_{12} - |g|/2, \quad b_{22} = c_{22}$$

The conditions of positive definiteness of (7.2) are easily obtained. The sufficient condition consists of the satisfaction of one of the group of inequalities

a) $b_{11} > 0, \quad b_{12} \geqslant 0, \quad b_{22} > 0$

b) $b_{11} > 0, \quad b_{12} < 0, \quad b_{11}b_{22} - b_{12}^2 > 0$

The problem of the resonance $\lambda_1 + 3\lambda_2 = m_*$ is solved differently; we have

$$H_4{}^0 = c_{11}r_1^2 + 2c_{12}r_1r_2 + c_{22}r_2^2 - |g|\, r_1^{1/2} r_2^{3/2} \tag{7.3}$$

As indicated above, the necessary condition for the function H_4^*, and hence also H_4°, to be positive definite is the positive definiteness of H_4. Hence the inequalities $c_{11} > 0$, $c_{22} > 0$ must necessarily be satisfied in (7.3). Under these conditions the function H_4° is positive definite along the straight lines $r_1 = 0$ and $r_2 = 0$. We introduce outside the straight line $r_1 = 0$ a new variable $y = r_2/r_1$. Then, when $r_1 \neq 0$, we have

$$H_4{}^\circ = r_1^2 f(y), \quad f(y) = c_{11} + 2c_{12}y + c_{22}y^2 - |g|\, y\sqrt{y}$$

Hence the necessary and sufficient condition for H_4^* to be positive definite is in this case the lack of a non-negative root of the equation $f(y) = 0$.

We have thus obtained the conditions for the function H_4^* to be positive definite for all cases of fourth-order resonance of a periodic system with two degrees of freedom. The conditions of negative definiteness are derived similarly. It will be seen that if the coefficients $c_{ij}\,(i, j = 1, 2)$ and g analytically depend on the parameter e, the form H_4^* is of fixed sign when $e = 0$, and when $g = 0$, and for fairly small $e \neq 0$ the form H_4^* is also of fixed sign.

According to the generalized Theorem 1, the satisfaction of these conditions for the function H_4^* to be of fixed sign ensures the Lyapunov stability of the system.

The autonomous system with three degrees of freedom. According to the generalized Theorem 2 the sufficient condition of Lyapunov stability for this system is the fixed sign of the form H_4^* on the linear manifold (5.1) when $n = 3$. As above, we shall limit ourselves to obtaining the conditions of positive definiteness of the function H_4^*. The conditions of negative definiteness are derived similarly.

All possible fourth-order resonances reduce in this system to two resonances

$$\lambda_1 + 3\lambda_2 = 0, \quad \lambda_1 + \lambda_2 + 2\lambda_3 = 0$$

For the first of these resonances the inequality $\lambda_1\lambda_2 < 0$ must necessarily be satisfied, and function H_4^* has the form

$$H_4{}^* = H_4(r_1, r_2, r_3) + g r_1^{1/2} r_2^{3/2} \cos \nu$$

Hence on manifold (5.1) we have

$$H_4{}^\circ = D(r_1, r_2) - |g|\, r_1^{1/2} r_2^{3/2} \tag{7.4}$$

$$D(r_1, r_2) = d_{11}r_1^2 + 2d_{12}r_1r_2 + d_{22}r_2^2 \tag{7.5}$$

$$d_{ii} = c_{ii} + c_{33}\frac{\lambda_i^2}{\lambda_3^2} - 2c_{i3}\frac{\lambda_i}{\lambda_3} \quad (i = 1, 2)$$

$$d_{12} = c_{12} - c_{13}\frac{\lambda_2}{\lambda_3} - c_{23}\frac{\lambda_1}{\lambda_3} + c_{33}\frac{\lambda_1\lambda_2}{\lambda_3^2}$$

The function (7.4) has the same form as (7.3). Hence the subsequent analysis of the fixed sign of H_4° is the same as the respective analysis of (7.3). However the supplementary condition

$$r_3 = -\frac{1}{\lambda_3}(\lambda_1 r_1 + \lambda_2 r_2) \geqslant 0$$

must be taken into account.

Let us summarize the results obtained.

The necessary and sufficient conditon for H_4^* to be positive definite on the manifold (5.1) is

$$f_0(y) - |g|\, y\sqrt{y} \neq 0, \quad f_0(y) = d_{11} + 2d_{12}y + d_{22}y^2$$

when $y \geqslant 3$, if $\lambda_1\lambda_3 > 0$, and $0 \leqslant y \leqslant 3$, if $\lambda_1\lambda_3 < 0$.

The analysis of resonance $\lambda_1 + \lambda_2 + 2\lambda_3 = 0$ is similar to the preceding. Depending on the sign of the constants $\lambda_1, \lambda_2, \lambda_3$, we obtain three groups of conditions.

Let us formulate the final result.

The function H_4° on the manifold (5.1) has the form

$$H_4^* = D(r_1, r_2) + |g| \left(\frac{\lambda_1}{\lambda_3} r_1^{3/2} r_2^{1/2} + \frac{\lambda_2}{\lambda_3} r_1^{1/2} r_2^{3/2} \right)$$

where $D(r_1, r_2)$ is determined by formulae (7.5). The necessary and sufficient condition for H_4^* to be positive definite on manifold (5.1) is

$$f_0(y) + |g| \left(\frac{\lambda_1}{\lambda_3} \sqrt{y} + \frac{\lambda_2}{\lambda_3} y \sqrt{y} \right) \neq 0$$

where, depending on the signs of the constants $\lambda_1, \lambda_2, \lambda_3$, the variable y varies within the limits

a) $\lambda_1\lambda_2 > 0, \quad y \geqslant 0$
b) $\lambda_1\lambda_2 < 0, \quad \lambda_1\lambda_3 > 0, \quad y \geqslant -\lambda_1/\lambda_2$
c) $\lambda_1\lambda_2 < 0, \quad \lambda_1\lambda_3 < 0, \quad 0 \leqslant y \leqslant -\lambda_1/\lambda_2$

8. The stability of triangular libration points of the restricted three-body problem. The statement of the problem in the linear approximation and, also, fundamental results of a non-linear analysis of this problem are given in /9/. Using these results, we shall confine the examinationtc two special cases.

The plane elliptic three-body problem. The system in this case has two degrees of freedom, and the equations of pertrubed motion have the form (1.1) with a Hamiltonian peridocially dependent on time. In addition, the Hamiltonian H depends on two parameters, μ, the ratio of the mass of one of the two attracting bodies to the sum of masses of both bodies, and the eccentricity e.

According to /9/, for a fairly small eccentricity and all values of μ from the interval (8.1)

$$0\ 02429433\ldots = \mu_1 < \mu < \mu^* = 0{,}0385208\ldots \tag{8.1}$$

except those corresponding to the resonances

$$\lambda_1 + 2\lambda_2 = 0, \; 3\lambda_1 + \lambda_2 = 0, \; 3\lambda_2 = -2 \tag{8.2}$$

in the normalized Hamiltonian, the form $H_3^* \equiv 0$, and the fourth-order form H_4^* has the form

$$H_4^* = c_{11}(\mu, e) r_1^2 + 2c_{12}(\mu, e) r_1 r_2 + c_{22}(\mu, e) r_2^2 + g(\mu, e) r_1^{|m_1|/2} r_2^{|m_2|/2} \cos v \tag{8.3}$$

When there are no fourth-order resonances, if $g(\mu, e) \equiv 0$, while on other resonance curves possible in the interval (8.1)

$$3\lambda_1 - \lambda_2 = 2, \; 3\lambda_1 - \lambda_2 = 3, \; \lambda_1 - 2\lambda_2 = 2, \; \lambda_1 - 3\lambda_2 = -1, \quad 4\lambda_1 = 3 \tag{8.4}$$

the function g depends only on e and vanishes when $e = 0$. The coefficients c_{ij} $(i, j = 1, 2)$ are also analytic functions of e, and when $e = 0$ they take positive values, and $c_{11}(\mu, 0) > 0.5$, $c_{12}(\mu, 0) > 10{,}0$, $c_{22}(\mu, 0) > 7.5$. Therefore for a fairly small eccentricity $0 \leqslant e \ll 1$ the function (8.3) is positive definite irrespective of the presence of resonances (8.4). The following theorem follows from Theorem 1 and its generalization.

Theorem 3. The triangular libration points of a plane elliptical restricted three-body problem for a fairly small eccentricity is Lyapunov stable for all values of μ from the interval (8.1), excluding those corresponding to resonances (8.2).

Remark. For μ belonging to the interval (8.1) and small e the resonance $3\lambda_2 = -2$ remained uninvestigated. It requires the analysis of terms of order e^2 and higher in coefficients of the normalized Hamiltonian. According to /9/ the resonances $\lambda_1 + 2\lambda_2 = 0$, $3\lambda_1 + \lambda_2 = 0$ result in instability.

For arbitrary non-resonance, in the sense of the lack of second-fourth order resonances, values of μ, e, as the result of the numerical investigation in /13/ in the stability region MS (Markeyev-Sokol'skii) regions were separated to a first approximation where the form (3.1) is of fixed sign (see also /9/, pp.68-169, Figs. 18 and 19). The application of Theorem 1 to these results establishes the following theorem.

Theorem 4. The triangular libration points of the plane elliptic restricted three-body problem are Lyapunov stable for all non-resonance parameters μ, e from MS regions.

The three dimensional circular three-body problem. In this problem the Hamiltonian does not explicitly depend on time and $n = 3$. It is shown in /9/ that for all μ from the intervals

$$0 < \mu < 0{,}010913\ldots;\ 0{,}016376\ldots < \mu < \mu_1 = 0{,}024293\ldots \qquad (8.5)$$
$$\mu_1 < \mu < 0{,}038520\ldots$$

system (5.3) has no other solution than the trivial. Hence we have the following theorem.

Theorem 5. The triangular libration points of the three-dimensional restricted three-body problem is Lyapunov stable for all values of the parameter μ from the interval (8.3).

REFERENCES

1. ARNOLD V.I., Stability of the Hamiltonian equilibrium position of the system of ordinary differential equations in the general elliptic case. Dokl. AN SSSR, Vol. 137, No. 2, 1961.
2. ARNOLD V.I., Small denominators and problems of the stability of motion in classical and celestial mechanics. Uspekhi Matem. Nauk, Vol.18, No.6, 1963.
3. MOSER J., On invariant curves of area preserving mapping of an annulus. Nachrit. Akad. Wiss. Göttingen, Mat. Phys., No.1, 1962.
4. MOSER J., Lectures on Hamiltonian Systems. Mem. Amer. Math. Soc. Vol.81, 1968.
5. MARKEYEV A.P., On the stability of a canonical system with two degrees of freedom in the presence of resonance. PMM, Vol.32, No.4, 1963.
6. MARKEYEV A.P., On the problem of the stability of the equilibrium position of Hamiltonian systems. PMM, Vol.34, No.6, 1970.
7. SOKOL'SKII A.G., On the stability of an autonomous Hamiltonian system with two degrees of freedom in the case of equal frequences. PMM, Vol.38, Nos.5, 1974.
8. BIRKHOFF D., Dynamical Systems. Moscow-Leningrad, Gostekhizdat, 1948.
9. MARKEYEV A.P., Libration Points in Celestial Mechanics and Cosmodynamics. Moscow Nauka, 1978.
10. POINCARÉ A., New Methods of Celestial Mechanics. Col. Works, Vol.2. Moscow, Nauka, 1972.
11. DARBOUX G., Sur le probleme de Pfaff. Bul. Mathem. Astron. Vol.6, 1882.
12. KUNITSYN A.L. and MARKEYEV A.P., Stability in Cases of Resonance. Itogi, Nauki i Tekhniki. Obshchaya Mekhanika. Vol.4, 1979.
13. MARKEYEV A.P. and SOKOL'SKII A.G., Numerical investigation of the Lagrangian solutions of the elliptic restricted three-body problem PMM, Vol.38, No.1, 1974.

Translated by J.J.D.

PMM U.S.S.R., Vol.49, No.3, pp. 281-289, 1985
Printed in Great Britain

0021-8928/85 $10.00+0.00
Pergamon Journals Ltd.

ESTIMATE OF THE STABILITY OF A DYNAMIC SYSTEM ON THE BASIS OF THE QUASISTATIONARITY PRINCIPLE *

YU.M. VOLIN

The following problem is formulated and solved: in what cases, and on what basis for examining the stability of the stationary solution of a "quasistationary" system can we judge the stability of the stationary solution of the initial system? The theorems which formulate the necessary and sufficient conditions of the stability are proved. It is shown how the results obtained can be used to examine the thermal stability of a chemical reactor.

1. Suppose it is required to examine the stability of the stationary state of a dynamic system. When using Lyapunov's first method this problem reduces (if we do not consider special cases) to the problem of verifying the stability of the zeroth solution of the linearized system. We will assume that the latter can be represented in the form

$$\frac{dy}{dt} = Ay + Bz, \quad \frac{dz}{dt} = Cy + Dz; \quad y \in R^m, \quad z \in R^l \qquad (1.1)$$

We will also introduce the notation $x = (y_1, \ldots, y_m, z_1, \ldots, z_l)^T$, $m + l = n$, where the index T denotes transposition.

**Prikl. Matem. Mekhan.*, 49, 3, 366-376, 1985